J. Fluid Mech. (1986), *vol.* 173, *pp.* 273–288
Printed in Great Britain

Secondary bifurcations of Taylor vortices into wavy inflow or outflow boundaries

By G. IOOSS
Université de Nice, I.M.S.P. Mathématiques, Parc Valrose, 06034 Nice Cedex, France

(Received 9 April 1986)

Experiments of Andereck *et al.* (1986) with corotating cylinders, show that Taylor-vortex flow (TVF) can bifurcate into one of the following cellular flows: wavy vortices (WV), twisted vortices (TW), wavy inflow boundaries (WIB), wavy outflow boundaries (WOB). We describe here the structure of these different flows, showing how they result from simple symmetry breaking. Moreover we consider the codimension-two situation where WIB and WOB interact, since this is an observed physical situation.

The method used in this paper is based on symmetry arguments. It differs notably from the Liapunov–Schmidt reduction used in particular by Golubitsky & Stewart (1986) on the same problem with counter-rotating cylinders. Here we take into account all the dynamics, instead of restricting the study to oscillating solutions. In addition to the standard oscillatory modes, we have a translational mode due to the indeterminacy of TVF under the shifts along the axis. We derive an amplitude-expansion procedure which allows the translational mode to depend on time. Our amplitude equations have nevertheless a simple structure because the oscillatory modes have a precise symmetry. They break, in general, the rotational invariance and they are either symmetric or antisymmetric with respect to the plane $z = 0$. Moreover, the most typical cases are when either of these modes has the same axial period as TVF or when their axial period is double this. This leads to four different cases which are shown to give WV, TW, WIB or WOB, all these flows being 'rotating waves', i.e. they are steady in a suitable rotating frame.

Finally we consider the interaction between WIB and WOB that occurs when, at the onset of instability, the two critical modes arise simultaneously. In this case we show in particular that there may exist a stable quasi-periodic flow bifurcating from WIB or WOB. The two main frequencies are those of underlying WIB and WOB, while there may exist a third frequency corresponding to a slow superposed travelling wave in the axial direction.

The method was used in the counter-rotating case for interacting non-axisymmetric modes (see Chossat *et al.* 1986). One of the original contributions here is not only to clarify the origin of all observed bifurcations from TVF, but also to handle the translational mode which may not stay small. This technique combined with centre-manifold and equivariance techniques may be helpful for many problems starting with orbits of solutions, such as the TVF considered here.

1. Introduction

Recent experiments by Andereck, Liu & Swinney (1986) on the Taylor–Couette problem for corotating cylinders have provided many details about when the Taylor-vortex flow (TVF) loses its stability. They were able to obtain wavy vortices, vortices with flat boundaries and internal waves (twists), and vortices with wavy

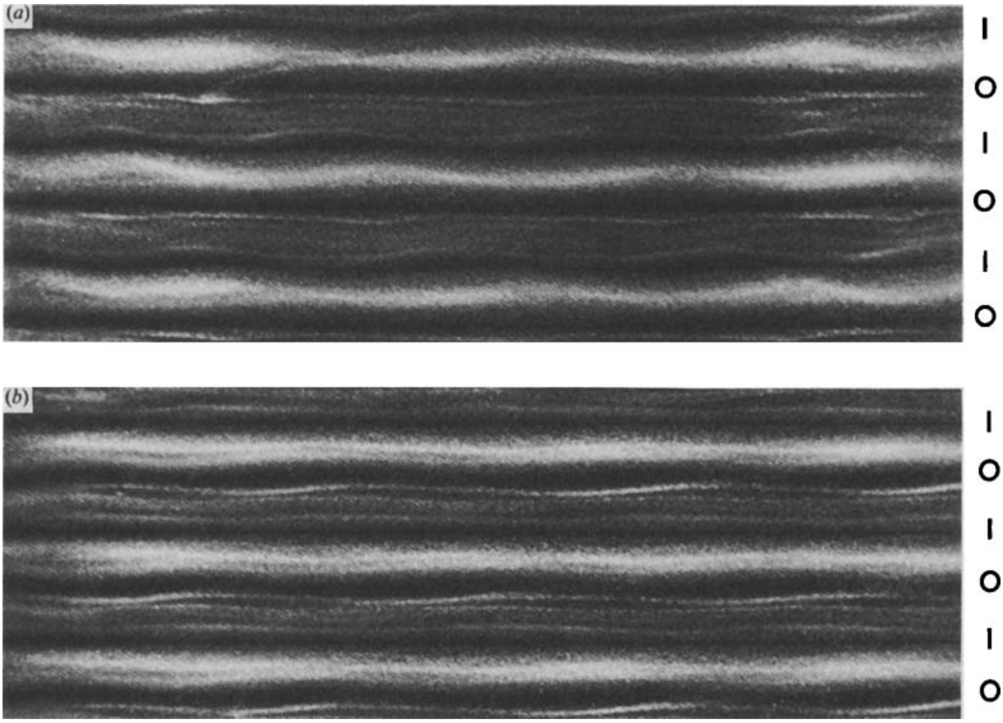

FIGURE 1. (a) WIB, (b) WOB from Andereck *et al.* (1986): O, outflow boundary; I, inflow boundary. boundary. Note that the axial period is twice the period of the TVF.

outflow boundaries (WOB) and wavy inflow boundaries (WIB). We are interested here in explaining the mathematical nature of the flows referred to as WOB and WIB. For a certain range of the angular velocity Ω_2 of the outer cylinder, one of these flows occurs when TVF disappears, when the angular velocity of the inner cylinder Ω_1 increases. For a different range of Ω_2 the other flow occurs. The existence of these flows also depends on the axial wavelength of the original TVF (Andereck *et al.* 1986). These flows are both rotating waves, i.e. in a suitable rotating frame they are steady. Moreover, for the WOB (WIB) the flat boundaries corresponding to an outward flow (inward flow) for TVF become wavy (see figures 1 and 2). For either flow, the axial wavelength of the pattern is twice that of the TVF. Let us note that recent numerical computations by Nagata (1986), in the small-gap case of almost corotating cylinders, show two types of subharmonic modes which in fact are WIB and WOB. Moreover Jones (1985) studied numerically the instability of TVF for $\Omega_2 = 0$, and found a 'subharmonic jet mode' which in fact is WOB.

The method used in this paper is mainly based on symmetry arguments. Even though all symmetries of the Taylor–Couette problem were known at the time of G. I. Taylor, it is only in the last eight years that these symmetries could be used in a systematic way, since each bifurcation corresponds to a symmetry breaking of the observed solution. Roughly speaking, the linearized stability analysis allows us to 'guess' the form of the bifurcating flow, and the nonlinear analysis (expansion technique) gives the amplitude and higher-order terms. In fact, when there are many symmetries, as here, the linearized stability analysis is not sufficient to give a qualitative idea of the bifurcated solution: there is too much choice! It is then necessary to make the nonlinear analysis more general, in order to avoid losing anything and to be able to decide what flow will finally be seen. We derive an

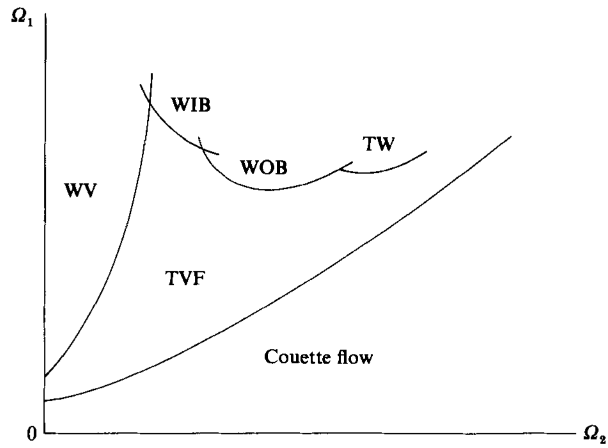

FIGURE 2. Experimental results for corotating cylinders; partial picture (see Andereck *et al.* 1986).

amplitude-expansion procedure of the same type as the classical ones (see Stuart 1971), which is fully justified by recent progress in the theory of dynamical systems and which is adapted here to the case of a one-parameter family of basic steady flows (TVF) leading to a 'translational mode'. This amplitude-expansion technique, which uses symmetry arguments, differs notably from the Liapunov–Schmidt reduction, used in particular by Golubitsky & Stewart (1986) on a similar problem. Here the great advantage is to be able to study the transients and to obtain unexpected solutions.

We show below how nature chooses between wavy vortices (WV), twisted vortices (TW), WIB and WOB. In the experiments of Andereck *et al.* (1986) there is a case where WIB and WOB interact. We study this codimension-two situation, and we show that each WIB and WOB bifurcates into a quasi-periodic flow which has basically the two main frequencies of WIB and WOB, and possibly a superposed low frequency corresponding to a slow axial travelling wave. This travelling wave has nothing to do with a global mean velocity upwards or downwards for particles, so we infer that this type of flow can be expected in finite cylinders provided that end effects are not too strong (as we change the boundary conditions in our analysis.) These quasi-periodic flows with three frequencies (one is very small) seem close to the so-called 'wavelets' of Andereck *et al.* (1986).

Let us finally note that there are many other possibilities for the bifurcation of TVF. They would correspond to flows with a multiple axial wavelength greater than $4\pi/\alpha$. Preliminary results are contained in Chossat & Iooss (1985). Such flows seem not to have yet been observed.

2. Notation and basic properties

Let us first specify the physical conditions that are taken into account for the mathematical formulation of the problem. We assume the fluid filling the domain between the two concentric cylinders to be viscous and incompressible, so that the equations driving the flow are the Navier–Stokes equations

$$\left.\begin{aligned} \frac{\partial V}{\partial t}+(V\cdot\nabla)\,V+\frac{1}{\rho}\nabla p &= \nu\Delta V, \\ \nabla\cdot V &= 0, \end{aligned}\right\} \tag{1}$$

where ρ is the constant volumic mass, ν is the kinematic viscosity, p is the pressure and V the velocity of fluid particles. System (1) holds for x belonging to a domain Ω defined in cylindrical coordinates by $R_1 < r < R_2$, $z \in \mathbb{R}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Components of $V(x, t)$ are written (v_r, v_θ, v_z) in cylindrical coordinates, so the boundary conditions are (neglecting top and bottom conditions for the moment)

$$v_r = v_z = 0, \quad v_\theta = \Omega_j R_j \quad \text{at } r = R_j, \quad j = 1, 2. \tag{2}$$

Now we put the system into dimensionless variables by choosing the following scales for length, time, pressure and velocity: R_1, Ω_1^{-1}, $\rho R_1^2 \Omega_1^2$, $R\Omega_1$ respectively, and the Reynolds number $\mathscr{R} = R_1^2 \Omega_1/\nu$. It is classical that the following velocity field, independent of (t, θ, z), is a solution of the non-dimensionalized equations

$$V^0 = (0, v^0, 0), \quad v^0 = \mathrm{A}r\left(1+\frac{B}{r^2}\right), \quad r \in \left[1, \frac{R_2}{R_1}\right], \tag{3}$$

with A and B depending on R_2/R_1 and Ω_2/Ω_1. This is Couette flow, which is observed in experiments when the Reynolds number is small enough. We now set

$$V = V^0 + U, \quad p = p^0 + q,$$

so that the system satisfied by (U, q) is the following:

$$\left.\begin{aligned} \frac{\partial U}{\partial t} + (U\cdot\nabla)\, V^0 + (V^0\cdot\nabla)\, U + (U\cdot\nabla)\, U + \nabla q &= \mathscr{R}^{-1}\Delta U, \\ \nabla\cdot U &= 0, \\ U|_{r=1,\, R_2/R_1} &= 0, \end{aligned}\right\} \tag{4}$$

where the parameters are $\mathscr{R}$, R_2/R_1, Ω_2/Ω_1.

The system (4) has the fundamental property of invariance under translations along the z-axis, reflections through the horizontal plane Oxy, and rotations about the z-axis. Let us elaborate this statement. In cylindrical coordinates we define three linear operators τ_a, S, R_ϕ which are representations of the translation $z \mapsto z+a$, the reflection $z \mapsto -z$ and the rotation about the axis $\theta \mapsto \theta + \phi$:

$$\left.\begin{aligned} [\tau_a U](r, \theta, z) &= U(r, \theta, z+a), \\ [\mathrm{S}U](r, \theta, z) &= (u_r(r, \theta, -z), u_\theta(r, \theta, -z), -u_z(r, \theta, -z)), \\ [\mathrm{R}_\phi U](r, \theta, z) &= U(r, \theta+\phi, z), \end{aligned}\right\} \tag{5}$$

and analogous relations for the scalar function q. We note that the Couette flow is invariant under τ_a, S and R_ϕ and that the system (4) commutes with these three linear operators (we say that it is equivariant under the three representations (5)).

Let us consider the cases when the TVF is a solution of (4). Here U_0, the TVF, is such that the plane $z = 0$ is an outflow boundary, and α is the axial wavenumber. We have the following properties:

$$\left.\begin{aligned} \mathrm{R}_\phi U_0 &= U_0 \quad \text{(TVF is axisymmetric)}, \\ \mathrm{S}U_0 &= U_0 \quad \text{(TVF is symmetric under the reflection } z \mapsto -z\text{)}, \\ \tau_{2\pi/\alpha} U_0 &= U_0 \quad \text{(axial periodicity)}. \end{aligned}\right\} \tag{6}$$

Moreover, it is clear that $\tau_a U_0$ is a TVF shifted by a along the z-axis, so for $a = \pi/\alpha$ (half of the period) the plane $z = 0$ is an inflow boundary for this TVF (see figure 3).

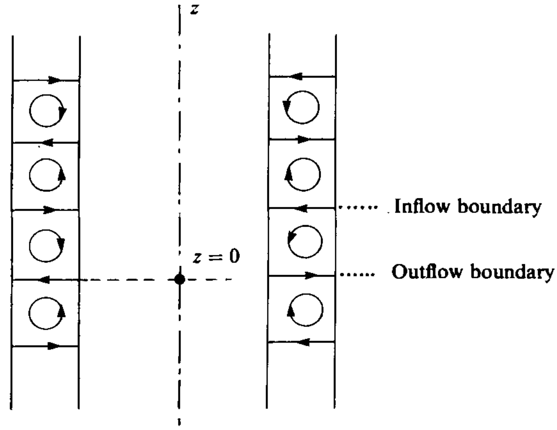

FIGURE 3. Taylor vortex flow (TVF), section view.

If we consider now the system satisfied by the perturbation of the basic flow U_0, this system (putting $V^0 + U_0$ instead of V^0 into (4)) is equivariant under the linear operators R_ϕ, S and $\tau_{2\pi/\alpha}$ (no longer under τ_a for any a). The linearized system will satisfy the same equivariance conditions.

Now, for mathematical purposes we replace the experimental boundary conditions on the top and the bottom of cylinders by a *periodicity condition*, the period being h. In fact, it is known physically that if we change the boundary conditions at the top and bottom of cylinders, we observe the same flow as before, provided that we are 'not too close' to these ends. Moreover, for rigid boundaries there is a singularity of the velocity field which impedes the mathematics. Now it is clear that we necessarily have an integer n such that

$$h = n2\pi/\alpha, \tag{7}$$

where n is the number of periods along the axis ($2n$ cells). Let us note that by fixing h we suppress the possibility of interactions between a continuum of wavenumbers (the linearized system has a continuous spectrum). Moreover it still allows many types of possible symmetry breaking for large n. By choosing $n = 1$ we would not be able to obtain any WOB or WIB.

The system (4) satisfied by the perturbation U of Couette flow may be written in a functional form as follows:

$$\frac{dU}{dt} = \mathscr{F}(\mu, U), \tag{8}$$

where $U(t)$ is a divergence-free vector function which satisfies homogeneous boundary conditions on the cylinders. The importance of (8) is that it avoids including the pressure term ∇q. This is done by projecting the system (4) on to a divergence-free space of vector fields with zero normal component on the cylinders. This space is orthogonal to ∇q with the L^2 scalar product. This is a classical way of showing that Navier–Stokes equations give a well-posed evolution system in a suitable Hilbert space, with a unique solution for the initial-value problem (see for instance Iooss 1984). We might consider μ in a three-dimensional space, by performing for instance on parameters $\mathscr{R}$, Ω_2/Ω_1, R_2/R_1. For simplicity of notation we shall first consider a single parameter μ (a function of the Reynolds number) and, for the codimension-two problem, another parameter ν (a function of Ω_2/Ω_1).

3. Linear stability analysis of the TVF

We have by construction, since U_0 is a TVF,

$$\mathscr{F}(\mu, U_0) = 0, \tag{9}$$

and due to the equivariance of $\mathscr{F}$

$$\mathscr{F}(\mu, \tau_a U_0) = 0 \quad \text{for any } a. \tag{10}$$

The stability of a TVF is governed by the eigenvalues of the linear operator

$$\mathscr{L}_{\mu,0} = \mathrm{D}_U \mathscr{F}(\mu, U_0). \tag{11}$$

Due to the equivariance of $\mathscr{F}$ we also have

$$\mathscr{F}(\mu, \tau_a U) = \tau_a \mathscr{F}(\mu, U) \quad \text{for any } U \text{ and } a. \tag{12}$$

Differentiating (12) with respect to U, and computing the result for $U = U_0$, we obtain

$$\mathscr{L}_{\mu,a} = \tau_a \mathscr{L}_{\mu,0} \tau_{-a}, \tag{13}$$

with

$$\mathscr{L}_{\mu,a} = \mathrm{D}_U \mathscr{F}(\mu, \tau_a U).$$

As a consequence, the eigenvalues of $\mathscr{L}_{\mu,0}$ are the same as those of $\mathscr{L}_{\mu,a}$ for any a.

Let us assume that the TVF becomes unstable in an oscillatory way. This means that, at criticality, $\pm i\omega_0$ are eigenvalues of $\mathscr{L}_{0,0}$ (if we define $\mu = 0$ as criticality). An eigenmode belonging to $i\omega_0$ has the following form:

$$\zeta_0 = \hat{U}(r, z)\, e^{im\theta}, \quad m \text{ integer}, \tag{14}$$

where $\hat{U}(r, z) = (\hat{u}_r(r, z), \hat{u}_\theta(r, z), \hat{u}_z(r, z))$. Since the linear operator $\mathscr{L}_{0,0}$ commutes with S, the vector function $\zeta_1 = \mathrm{S}\zeta_0$ is also an eigenmode, and we have to study whether ζ_1 and ζ_0 are colinear or not. In the first case, $\pm i\omega_0$ will be simple eigenvalues, while in the second case they will be double.

The crucial starting point of the nonlinear analysis rests on the behaviour of ζ_0 and ζ_1 under the group actions R_ϕ, S, $\tau_{2\pi/\alpha}$.

The actions of R_ϕ and S are easily expressed, since we have from (14)

$$\mathrm{R}_\phi \zeta_j = e^{im\phi}\zeta_j, \quad j = 0, 1; \quad \mathrm{S}\zeta_0 = \zeta_1, \quad \mathrm{S}\zeta_1 = \zeta_0. \tag{15}$$

Now the action of the operator $\tau_{2\pi/\alpha}$ on the eigenspace spanned by ζ_0 and ζ_1 has to satisfy the property that $[\tau_{2\pi/\alpha}]^n$ is the identity operator. This is due to the axial h-periodicity, n being the number of sub-periods for U. As a consequence the eigenvalues of $\tau_{2\pi/\alpha}$ on the space $\{\zeta_0, \zeta_1\}$ are nth roots of unity.

Let us choose ζ_0 such that

$$\tau_{2\pi/\alpha}\zeta_0 = e^{2i\pi l/n}\zeta_0, \quad l \text{ integer}; \tag{16}$$

then ζ_0 has again the form (14), and since we have the following identity:

$$\mathrm{S}\tau_a = \tau_{-a}\mathrm{S}, \tag{17}$$

we can deduce easily that

$$\tau_{2\pi/\alpha}\zeta_1 = \tau_{2\pi/\alpha}\mathrm{S}\zeta_0 = \mathrm{S}\tau_{-2\pi/\alpha}\zeta_0 = e^{-2i\pi l/n}\mathrm{S}\zeta_0 = e^{-2i\pi l/n}\zeta_1.$$

It follows that if $e^{-2i\pi/n} \neq e^{2i\pi l/n}$ (i.e. if $l/n \neq 0, \frac{1}{2}$), the eigenspaces belonging to $\pm i\omega_0$ are in general two-dimensional. Now, for $l/n = 0$ or $\frac{1}{2}$ the eigenvalues $\pm i\omega_0$ are in general *simple*.

Remark 1. For $l/n = l_0/n_0$, $n_0 \geqslant 3$, this leads to bifurcating solutions with an axial wavelength equal to $n_0 2\pi/\alpha = n_0 h/n$. Preliminary results are contained in Chossat & Iooss (1985). We do not consider such cases here since there is no experimental evidence of such symmetry breaking.

From now on let ζ_+ or ζ_- denote the eigenmode ζ_0 depending on whether we have

$$\mathrm{S}\zeta_+ = \zeta_+ \quad \text{or } \mathrm{S}\zeta_- = -\zeta_-. \tag{18}$$

These are the only possibilities since the eigenspace is one-dimensional and $\mathrm{S}^2 = \mathrm{Id}$. We have the following:

LEMMA 1. *For the linearized operator $\mathscr{L}_{0,\pi/\alpha}$ around the shifted TVF $\tau_{\pi/\alpha} U_0$, the eigenmode belonging to the eigenvalue $\mathrm{i}\omega_0$ is $\tau_{\pi/\alpha}\zeta_0$. If we have $l/n = \frac{1}{2}$, i.e. when we have*

$$\tau_{2\pi/\alpha}\zeta_0 = -\zeta_0, \tag{19}$$

then $\tau_{\pi/\alpha}$ exchanges even and odd eigenmodes:

$$\mathrm{S}(\tau_{\pi/\alpha}\zeta_+) = -(\tau_{\pi/\alpha}\zeta_+), \quad \mathrm{S}(\tau_{\pi/\alpha}\zeta_-) = (\tau_{\pi/\alpha}\zeta_-). \tag{20}$$

Indeed $\tau_{\pi/\alpha}\zeta_0$ is an eigenvector of $\mathscr{L}_{0,\pi/\alpha}$, due to the identity (13) with $a = \pi/\alpha$. Now, by using the identities (17) and (19) we can write

$$\mathrm{S}(\tau_{\pi/\alpha}\zeta_\pm) = \tau_{-\pi/\alpha}\mathrm{S}\zeta_\pm = \tau_{\pi/\alpha}\tau_{-2\pi/\alpha}\mathrm{S}\zeta_\pm = -\tau_{\pi/\alpha}\mathrm{S}\zeta_\pm,$$

and (18) leads to (20).

Finally, it is important to note that 0 is also an eigenvalue, owing to the translational invariance of the system (8).

If we differentiate (10) with respect to a, at $a = 0$ we get

$$\mathrm{D}_U\mathscr{F}(\mu, U_0)\cdot\xi_0 = 0, \tag{21}$$

where

$$\xi_0 = \frac{\mathrm{d}}{\mathrm{d}a}\tau_a U_0|_{a=0}$$

is the eigenvector belonging to the 0 eigenvalue. It follows simply that

$$\mathrm{R}_\phi\xi_0 = \xi_0, \quad \tau_{2\pi/\alpha}\xi_0 = \xi_0, \tag{22}$$

and

$$\mathrm{S}\xi_0 = -\xi_0 \tag{23}$$

result from the differentiation of the identity

$$\mathrm{S}\tau_a U_0 = \tau_{-a} U_0 \tag{24}$$

with respect to a at $a = 0$.

4. Amplitude equations

The usual way to define amplitudes is to set

$$U - U_0 = C\xi_0 + A\zeta_0 + \bar{A}\bar{\zeta}_0 + V, \tag{25}$$

where we can look for $V = \Phi(\mu, A, \bar{A}, C)$ in a complementary space of the three-dimensional space spanned by $\{\xi_0, \zeta_0, \bar{\zeta}_0\}$. The function Φ defines a so-called centre manifold in the space of vector functions where U lies. This manifold is locally invariant under (8), i.e. if one starts on it, one stays on it provided that one stays in some neighbourhood of U_0. Moreover, it is locally attracting, i.e. the long-time dynamics in a neighbourhood of U_0 is governed by the restriction of (8) on to this manifold. This new system is here three-dimensional and retains all the equivariances of the original system propagated via the definition (25) of the variables A, $\bar{A}$, C. This three-dimensional differential equation is in fact a particular case of the amplitude equations, known for a long time by fluid dynamicists working on nonlinear stability problems (Stuart 1971). The mathematical theory of centre manifolds may be found, for instance, in Marsden & McCracken (1976) or Iooss (1979) amongst many others; and in the presence of symmetries in Ruelle (1973).

In fact, we must realize that (25) is only valid in a neighbourhood of $\boldsymbol{U} = \boldsymbol{U}_0$, which is unhelpful if the flow we are looking for moves in a neighbourhood of the full orbit $\tau_a \boldsymbol{U}_0$, $a \in (0, 2\pi/\alpha)$ which represents all possible TVFs. A better formulation, which allows this last possibility, is to set

$$\boldsymbol{U} = \tau_C(\boldsymbol{U}_0 + A\zeta_0 + \bar{A}\bar{\zeta}_0 + \boldsymbol{V}), \tag{26}$$

where $\boldsymbol{V}$ lies in the same complementary space as for (25).

The difficulty here is that the decomposition (26) is nonlinear, but it is valid for large C and allows us the possibility of using the translational equivariance of (8). This decomposition was not used in Chossat & Iooss (1985), so they could not obtain any bifurcated solutions other than rotating waves, with a constant C. Even though (26) is a nonlinear decomposition of $\boldsymbol{U}$, we note that for C close to 0, we recover (25) up to second-order terms since we have, thanks to (21),

$$\tau_C = \mathrm{Id} + O(C),$$
$$\tau_C \boldsymbol{U}_0 = \boldsymbol{U}_0 + C\xi_0 + O(C^2).$$

Replacing (26) in (8), we find immediately by using (12) that

$$\left(\xi_0 + A\frac{\partial \zeta_0}{\partial z} + \bar{A}\frac{\partial \bar{\zeta}_0}{\partial z} + \frac{\partial \boldsymbol{V}}{\partial z}\right)\frac{\mathrm{d}C}{\mathrm{d}t} + \frac{\mathrm{d}A}{\mathrm{d}t}\zeta_0 + \frac{\mathrm{d}\bar{A}}{\mathrm{d}t}\bar{\zeta}_0 + \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \mathscr{F}(\mu, \boldsymbol{U}_0 + A\zeta_0 + \bar{A}\bar{\zeta}_0 + \boldsymbol{V}). \tag{27}$$

We look now for a centre manifold such that

$$\boldsymbol{V} = \boldsymbol{\Phi}(\mu, A, \bar{A}) \tag{28}$$

since in (27) C has disappeared.

The amplitude equations are now in the following form:

$$\left.\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}t} &= h(\mu, A, \bar{A}) \quad \text{(real)}, \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= f(\mu, A, \bar{A}) \quad \text{(complex)}, \end{aligned}\right\} \tag{29}$$

on which we have to propagate the equivariances of (8). By using (15), (17) and (18), the actions of τ_a, S, R_ϕ lead immediately to the following properties, due to the definition (26) of A and C:

$$\left.\begin{aligned} \tau_a&: \quad A \mapsto A, \quad C \mapsto C + a, \\ \mathrm{R}_\phi&: \quad A \mapsto A\,\mathrm{e}^{\mathrm{i}m\phi}, \quad C \mapsto C, \\ \mathrm{S}&: \quad C \mapsto -C, \quad A \mapsto A \quad \text{if } \zeta_0 = \zeta_+, \quad A \mapsto -A \quad \text{if } \zeta_0 = \zeta_-. \end{aligned}\right\} \tag{30}$$

The equivariance of (29) under the representations of τ_a, S, R_ϕ defined by (30) gives us an *a priori* form for f and h. We first note that h is identically 0. This is obvious by combining the equivariances under S and $R_{\pi/m}$. Finally it is clear that (30) leads to amplitude equations of the form

$$\frac{dC}{dt} = 0, \tag{31a}$$

$$\frac{dA}{dt} = Ag(\mu, |A|^2) = (i\omega_0 + a\mu)\,A + bA|A|^2 + \text{h.o.t.} \tag{31b}$$

where h.o.t. means higher-order terms. The factor $i\omega_0 + a\mu$ in the linear part is the principal part of the eigenvalue which perturbs the critical eigenvalue $i\omega_0$. We may observe that we are reduced to a standard 'Hopf bifurcation', since (31*b*) is simply the Landau amplitude equation.

5. Wavy vortices, twisted vortices, WIB and WOB

The zero solution in (31), gives a TVF (see (26)). Now we clearly have the following other solution of (31):

$$A = \rho_0\, e^{i(\Omega_0 t + \phi_0)}, \quad C = \text{const}, \tag{32}$$

with $a_r\mu + b_r\rho_0^2 + \ldots = 0$, i.e.

$$\left.\begin{aligned} \rho_0 &= \left(-\frac{a_r\mu}{b_r}\right)^{\frac{1}{2}} + O(|\mu|^{\frac{3}{2}}), \\ \Omega_0 &= \omega_0 + a_i\,\mu + b_i\,\rho_0^2 + \ldots, \end{aligned}\right\} \tag{33}$$

where the subscripts r and i respectively mean real part and imaginary part. This family of solutions gives the following velocity field:

$$\boldsymbol{U} = \boldsymbol{U}_0(r, z+C) + \rho_0\{\hat{\boldsymbol{U}}(r, z+C)\, e^{i(m\theta + \Omega_0 t + \phi_0)} + \text{c.c.}\} + \text{h.o.t.} \tag{34}$$

As a consequence of the analysis, this flow has the structure of *rotating waves*. In fact, due to (30) and

$$\arg(A(t)) = \Omega_0 t + \text{const},$$

we have for the full velocity field (not only for the principal part)

$$\boldsymbol{U}(t) = R_{\Omega_0 t/m}\, \boldsymbol{U}|_{t=0}. \tag{35}$$

This means that the flow is steady if we rotate the reference frame with a rotation rate Ω_0/m around the axis.

If $\tau_{2\pi/\alpha}\zeta_0 = \zeta_0$ we observe that $\tau_{2\pi/\alpha}$ reduces to the identity on A and C (see (26)). This means that in this case $\boldsymbol{U}$ in (34) is finally $2\pi/\alpha$-periodic in z.

If $\tau_{2\pi/\alpha}\zeta_0 = -\zeta_0$, then $\tau_{2\pi/\alpha}$ acts as follows: $A \mapsto -A$, $C \mapsto C$ (see (26)). This means that in this case, shifting z by $2\pi/\alpha$ is the same as rotating θ by π/m or translating time by π/Ω_0. Moreover, since $\tau_{4\pi/\alpha}$ is the identity on (A, C), the flow (34) will be $4\pi/\alpha$-periodic in z. Hence, in this case the axial period doubles.

Now, if $\zeta_0 = \zeta_+$, the flow (34) for $C = 0$ is invariant under S: $S\boldsymbol{U} = \boldsymbol{U}$. This property is true on the principal part of (34), so it is also true for the complete expression of $\boldsymbol{U}$, due to the propagation of this invariance on the $\boldsymbol{V}$-part defined by (26). This implies that $u_z = 0$ on the plane $z = 0$, as for TVF. This property is also valid on each plane half of the axial period apart. For instance, if the axial period is $2\pi/\alpha$, we have $u_z|_{z=\pi/\alpha} = u_z|_{z=-\pi/\alpha}$ while the invariance under S implies $u_z|_{z=\pi/\alpha} = -u_z|_{z=-\pi/\alpha}$, hence $u_z|_{z=\pi/\alpha} = 0$.

Finally, if $\zeta_0 = \zeta_-$, the plane $z = 0$ has no reason to be such that $u_z = 0$, and this cell boundary is then wavy. Let us show that if $\tau_{2\pi/\alpha}\zeta_0 = -\zeta_0$ then the boundary

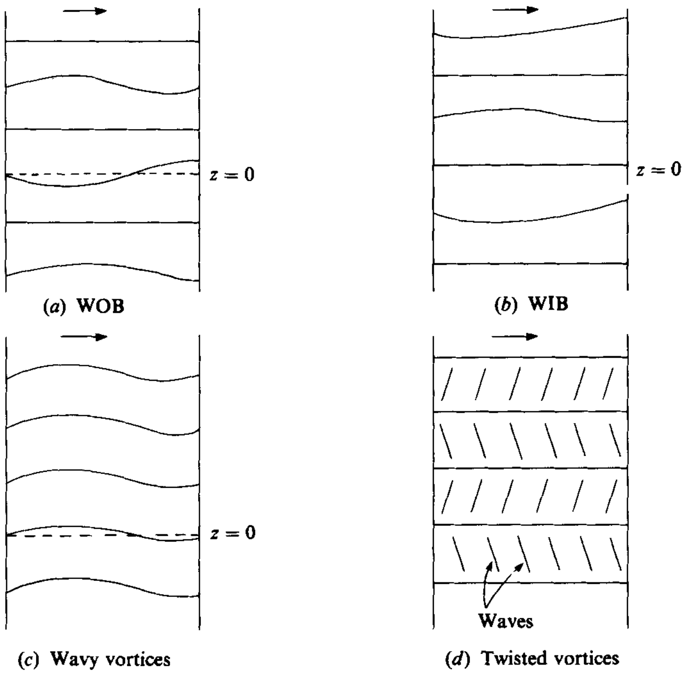

FIGURE 4. (a, b). Wavy outflow or inflow boundaries (front view); (c, d) wavy vortices and twisted vortices. All these flows are rotating waves ($\rightarrow$).

$z = \pi/\alpha$ stays flat (the flow has an axial period of $4\pi/\alpha$). In fact, if we started the analysis with $U_0' = \tau_{\pi/\alpha} U_0$ instead of U_0, we could make the same idea work, using $\zeta_+' = \tau_{\pi/\alpha}\zeta_-$ (see (20)). We necessarily recover the solution (34) shifted by $\tau_{2\pi/\alpha}$. Hence we have, for $C = 0$:

$$\tau_{\pi/\alpha} U = U_0' + \rho_0 \{\hat{U}' \, \mathrm{e}^{\mathrm{i}(m\theta + \Omega_0 t + \phi_0)} + \text{c.c.}\} + \text{h.o.t.},$$

with a symmetric $\hat{U}'$, and

$$\mathrm{S}(\tau_{\pi/\alpha} U) = \tau_{\pi/\alpha} U.$$

By the same argument as above, $u_z = 0$ on the plane $z = 0$ for the flow $\tau_{\pi/\alpha} U$, i.e. on the plane $z = \pi/\alpha$ for the flow U.

Let us now collect all these results for each possible case:

(i) $\mathrm{S}\zeta_0 = \zeta_0$, $\tau_{2\pi/\alpha}\zeta_0 = \zeta_0$, twisted vortices.

The flow has flat cells, π/α apart. The axial periodicity is $2\pi/\alpha$ (see figure 4).

(ii) $\mathrm{S}\zeta_0 = -\zeta_0$, $\tau_{2\pi/\alpha}\zeta_0 = \zeta_0$: wavy vortices.

Cells are wavy, axial periodicity is $2\pi/\alpha$ (see figure 4).

(iii) $\mathrm{S}\zeta_0 = \zeta_0$, $\tau_{2\pi/\alpha}\zeta_0 = -\zeta_0$: wavy inflow boundaries.

We took the convention that U_0 is the TVF with $z = 0$ as an outflow boundary. As a consequence, all the outflow boundaries stay flat and all inflow boundaries become wavy. The axial period is $4\pi/\alpha$ (see figure 4).

(iv) $\mathrm{S}\zeta_0 = -\zeta_0$, $\tau_{2\pi/\alpha}\zeta_0 = -\zeta_0$; wavy outflow boundaries.

All the inflow boundaries stay flat while the outflow boundaries of the TVF become wavy. The axial period is $4\pi/\alpha$ (see figure 4). Note that this flow was obtained numerically, using a direct computation, by Jones (1985) when the outer cylinder is at rest ('subharmonic jet mode').

Finally, we might observe that these flows are stable if they bifurcate supercritically, i.e. when $b_r < 0$ (since $a_r > 0$), and unstable if $b_r > 0$.

Remark 2. Given a numerically computed TVF, we would have to compute the eigenmode ζ_0 at criticality and then adapt the method developed in Demay & Iooss (1985) to compute coefficients a and b in (31).

6. Interaction between WIB and WOB

In the experiments of Andereck *et al.* (1986) there is a critical value of $\mathscr{R}$ and Ω_2/Ω_1 (for $R_1/R_2 = 0.883$) such that TVF becomes simultaneously unstable via two distinct ways leading to WIB and WOB. This means that at criticality we have two pairs of simple eigenvalues $\pm i\omega_+$, $\pm i\omega_-$ with eigenmodes $\zeta_+, \bar{\zeta}_+, \zeta_-, \bar{\zeta}_-$ such that

$$\zeta_\pm = \hat{U}_\pm(r,z)\,e^{im_\pm\theta}, \quad S\zeta_\pm = \pm\zeta_\pm, \quad \tau_{2\pi/\alpha}\zeta_\pm = -\zeta_\pm. \tag{36}$$

By convention, we take $m_\pm > 0$, hence $\omega_\pm$ may be < 0. We can perform the same type of analysis as above, by adding an additional parameter ν (a function of Ω_2/Ω_1). We start with the following decomposition which defines the amplitudes A, B, C:

$$U = \tau_C(U_0 + A\zeta_+ + \bar{A}\bar{\zeta}_+ + B\zeta_- + \bar{B}\bar{\zeta}_- + V), \tag{37}$$

with $V = \Phi(\mu, \nu, A, \bar{A}, B, \bar{B})$ in a complementary space of the five-dimensional space spanned by $\{\xi_0, \zeta_\pm, \bar{\zeta}_\pm\}$. By the same argument as above, the amplitude equations may be written as follows:

$$\left.\begin{aligned} \frac{dC}{dt} &= h(\mu, \nu, A, \bar{A}, B, \bar{B}), \\ \frac{dA}{dt} &= f(\mu, \nu, A, \bar{A}, B, \bar{B}), \\ \frac{dB}{dt} &= g(\mu, \nu, A, \bar{A}, B, \bar{B}). \end{aligned}\right\} \tag{38}$$

We have to propagate the equivariances of (8) taking account of (36). Hence we have the following representations of τ_a, R_ϕ, S, $\tau_{2\pi/\alpha}$:

$$\tau_a: \quad A \mapsto A, \quad B \mapsto B, \quad C \mapsto C + a, \tag{39a}$$

$$R_\phi: \quad A \mapsto A\,e^{im_+\phi}, \quad B \mapsto B\,e^{im_-\phi}, \quad C \mapsto C, \tag{39b}$$

$$S: \quad A \mapsto A, \quad B \mapsto -B, \quad C \mapsto -C, \tag{39c}$$

$$\tau_{2\pi/\alpha}: \quad A \mapsto -A, \quad B \mapsto -B, \quad C \mapsto C. \tag{39d}$$

Let us introduce n_+ and n_- such that

$$m_+ = d_0 n_+, \quad m_- = d_0 n_-, \tag{40}$$

where d_0 is the greatest common divisor of m_+ and m_-. It is then not difficult to prove the following properties (see Chossat, Demay & Iooss (1986) for the method to prove such properties):

LEMMA 2. *The amplitude equations in* (38) *satisfy*

$$\text{(i)}\qquad \left.\begin{aligned} f &= e^{i\psi_+} f_0(\rho_+, \rho_-, \theta),\\ g &= e^{i\psi_-} g_0(\rho_+, \rho_-, \theta),\\ h &= h_0(\rho_+, \rho_-, \theta), \end{aligned}\right\} \tag{41}$$

where $A = \rho_+ e^{i\psi_+}$, $B = \rho_- e^{i\psi_-}$, $\theta = n_+\psi_- - n_-\psi_+$, *and* f_0, g_0 *are* π*-periodic in* θ;

(ii) *If* n_+ *or* n_- *is even, then* $h \equiv 0$;

(iii) *If* n_+ *and* n_- *are odd then* h *is changed into* $-h$ *when* θ *is changed into* $\theta+\pi$.

If we expand f, g, h in powers of A, $\bar{A}$, B, $\bar{B}$ and then use (39), it is easy to see that the principal parts are as follows:

$$\frac{dA}{dt} = (i\omega_+ + a_1\mu + a_2\nu)A + bA|A|^2 + cA|B|^2 + d\bar{A}^{2n_- -1}B^{2n_+} + \text{h.o.t.}, \tag{42a}$$

$$\frac{dB}{dt} = (i\omega_- + a_1'\mu + a_2'\nu)B + b'B|A|^2 + c'B|B|^2 + d'A^{2n_-}\bar{B}^{2n_+ -1} + \text{h.o.t.}, \tag{42b}$$

$$\frac{dC}{dt} = \begin{cases} 0 & \text{if } n_- n_+ \text{ even},\\ eA^{n_-}\bar{B}^{n_+} + \bar{e}\bar{A}^{n_-}B^{n_+} + \text{h.o.t.} & \text{if } n_- n_+ \text{ odd}. \end{cases} \tag{42c}$$

The system (42*a*, *b*) without the d-terms is a very well-known system whose dynamics has been studied in different work (see Guckenheimer & Holmes (1983) and references therein).

Remark 3. It is clear from (42) that if n_+ or n_- is not 1, then the terms of order $2(n_+ + n_-) - 1$ occur in the h.o.t.

We notice from (42) that we recover WIB and WOB. In fact a WIB is obtained with

$$\left.\begin{aligned} B = 0, \quad C = \text{const}, \quad A &= \rho_0 e^{i(\Omega_0 t + \phi_0)}\\ a_{1r}\mu + a_{2r}\nu + b_r\rho_0^2 &= 0,\\ \Omega_0 = \omega_+ + a_{1i}\mu + a_{2i}\nu + b_i\rho_0^2. \end{aligned}\right\} \tag{43}$$

This family of solutions of (42) corresponds to a WIB by using the same arguments as in §5. In the same way we obtain a WOB with

$$\left.\begin{aligned} A = 0, \quad C = \text{const}, \quad B &= \rho_1 e^{i(\Omega_1 t + \phi_1)},\\ a_{1r}'\mu + a_{2r}'\nu + c_r'\rho_1^2 &= 0,\\ \Omega_1 = \omega_- + a_{1i}'\mu + a_{2i}'\nu + c_i'\rho_1^2. \end{aligned}\right\} \tag{44}$$

7. Secondary bifurcations of the WIB and WOB

Before studying the stability of the WIB and WOB, let us first note that if n_- is even, we could start the analysis with the shifted TVF $U_0' = \tau_{\pi/\alpha} U_0$ and, thanks to (20), we could exchange indices + and −. Hence we only have two different cases (n_+ and n_- have no common divisor):

(i) n_+ or n_- is even,

(ii) n_+ and n_- are odd.

Examples. If $|m_+ - m_-| = 1$, then $n_+ = m_+$, $n_- = m_-$ and case (i) applies. If $m_+ = m_-$ case (ii) applies since $n_+ = n_- = 1$.

To study the stability of the WIB let us set

$$\left.\begin{aligned} A &= (\rho_0+\rho')\, e^{i(\Omega_0 t+\phi)}, \\ B &= B'\, e^{i(n_-/n_+)(\Omega_0 t+\phi)}. \end{aligned}\right\} \tag{45}$$

Then, thanks to the lemma of §6, the system in $d\rho'/dt$, $d\phi/dt$, dB'/dt, dC/dt does not contain t and ϕ. In particular this avoids the need to compute Floquet exponents. The principal part of the linearized equations becomes

$$\left.\begin{aligned} \frac{d\rho'}{dt} &= 2b_r\rho_0^2\rho', \\ \frac{d\phi}{dt} &= 2b_i\rho_0\rho', \\ \frac{dC}{dt} &= \begin{cases} 0 & \text{if } n_+ \neq 1 \text{ or } n_- \text{ even}, \\ \rho_0^{n_-}(e\bar{B}'+\bar{e}B') & \text{if } n_+ = 1 \quad \text{and } n_- \text{ odd}, \end{cases} \\ \frac{dB'}{dt} &= [i(\omega_- - \omega_+)+\mu(a_1'-a_1)+\nu(a_2'-a_2)+(b'-b)\rho_0^2]\,B' + \begin{cases} 0 & \text{if } n_+ \neq 1 \\ d'\rho_0^{2n_-}\bar{B}' & \text{if } n_+ = 1. \end{cases} \end{aligned}\right\} \tag{46}$$

We then obtain 0 as a double eigenvalue (indeterminacies under time shifts and phase shift on A in (43)), and the classical real eigenvalue $2b_r\rho_0^2$, < 0 if the bifurcation of WIB is supercritical. There are two other complex eigenvalues λ, $\bar{\lambda}$ such that

$$\lambda = i(\omega_- - \omega_+)+\mu(a_1'-a_1)+\nu(a_2'-a_2)+(b'-b)\rho_0^2+O(|\mu|+|\nu|)^2. \tag{47}$$

In the parameter plane (μ, ν) we see that $\operatorname{Re}\lambda$ changes its sign when we cross the following line (using (43)):

$$0 = b_r^{-1}[\mu(a_{1r}'b_r - a_{1r}b_r') + \nu(a_{2r}'b_r - a_{2r}b_r')] \approx \operatorname{Re}\lambda = 0. \tag{48}$$

Crossing this line corresponds to a Hopf bifurcation on the system in (ρ', B') in a three-dimensional space. So there appears a new frequency Ω_H close to $|\omega_- - \omega_+|$ for A and B in (45). In fact we can say that the two basic frequencies for (A, B) are close to $|\omega_+|$ and $|\omega_-|$, since any integer combination of the basic frequencies is also a frequency of this bifurcating quasi-periodic flow.

If we now consider the amplitude C, we have two different cases. If n_+ or n_- is even, then C is constant and the bifurcated flow has just two frequencies with the additional property that if we rotate the frame suitably around the z-axis, we only see a periodic flow. Now, if n_+ and n_- are odd, then dC/dt is periodic with the frequency Ω_H. Its mean value κ is not necessarily 0. In fact, it is not hard to check that

$$\kappa = O(\rho_0^{n_-}\epsilon), \tag{49}$$

where ϵ is the distance between $\operatorname{Re}\lambda$ and the line (48). For the velocity field this corresponds to a slow travelling wave in the axial direction (velocity $= \kappa$), hence the bifurcating quasi-periodic flow has 3 frequencies in these cases (only one frequency in a suitable frame moving up with velocity κ and turning to match one of the rotating waves).

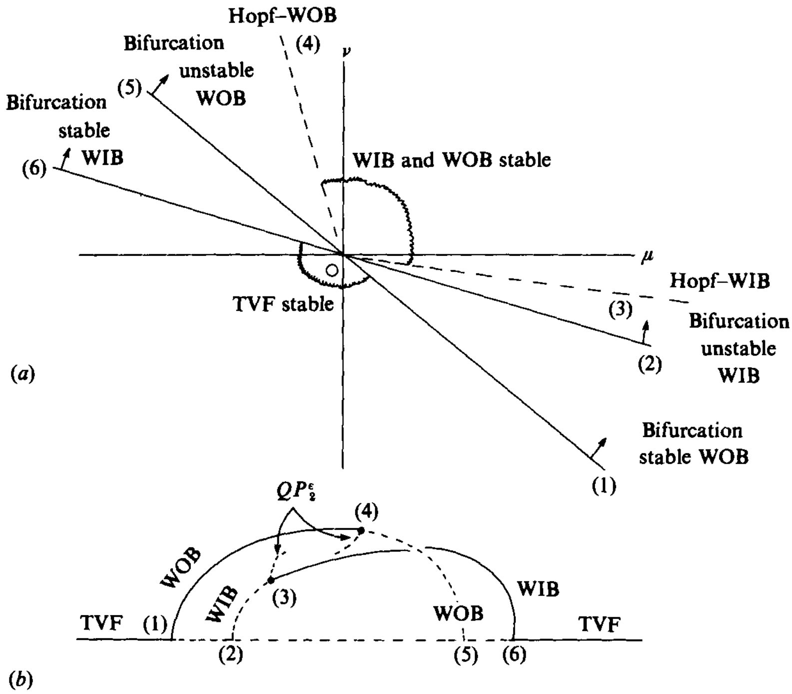

FIGURE 5. The first example of WIB–WOB interaction. (*a*) A parameter plane where we indicate the curves when bifurcations occur (QP_2^ϵ, wavelets on a not necessarily unique branch). (*b*) A bifurcation diagram (parameter, angle around 0 in (μ, ν)-plane). Here the WIB and WOB are both stable for a range of parameters.

In the same way, we can study the stability of the WOB. The eigenvalues are 0, which is double, $2c_r'\rho_1^2$ (< 0 if the bifurcation is supercritical), and two complex eigenvalues λ, $\bar{\lambda}$ such that

$$\lambda = \mathrm{i}(\omega_+ - \omega_-) + \mu(a_1 - a_1') + \nu(a_2 - a_2') + (c - c')\rho_1^2 + \text{h.o.t.} \tag{50}$$

A bifurcation into a quasi-periodic flow occurs, in the same way as above when (μ, ν) crosses the line

$$0 = c_r'^{-1}[\mu(a_{1r}c_r' - a_{1r}'c_r) + \nu(a_{2r}c_r' - a_{2r}'c_r)] \approx \operatorname{Re}\lambda = 0. \tag{51}$$

Remark 4. The principal part of the quasi-periodic flows we have just obtained is a pure superposition of two azimuthal wavenumbers m_+ and m_-. Each wave rotates alone on the inflow or on the outflow boundaries with a frequency close to ω_+ or ω_-. It is reasonable to suggest that these flows enter into the class of the so-called 'wavelets' of Andereck *et al.* (1986).

Remark 5. The computation of the amplitude equations (42) is based on the group action (39). If we consider the following other interactions, then the action of $\tau_{2\pi/\alpha}$ has to be changed:

(i) twisted vortices and WOB: $\tau_{2\pi/\alpha} \equiv \mathrm{S}$ on the variables A, B, C;
(ii) WIB and wavy vortices: $\tau_{2\pi/\alpha}: A \mapsto -A$, $B \mapsto B$, $C \mapsto C$;
(iii) twisted vortices and wavy vortices: $\tau_{2\pi/\alpha} = \mathrm{Id}$ on A, B, C.

In fact, these modifications of (39*d*) do not affect the analysis very much, which therefore remains mainly valid. It seems that, in the experiments of Andereck *et al.* (1986), interactions (i) and (ii) occur (see figure 2).

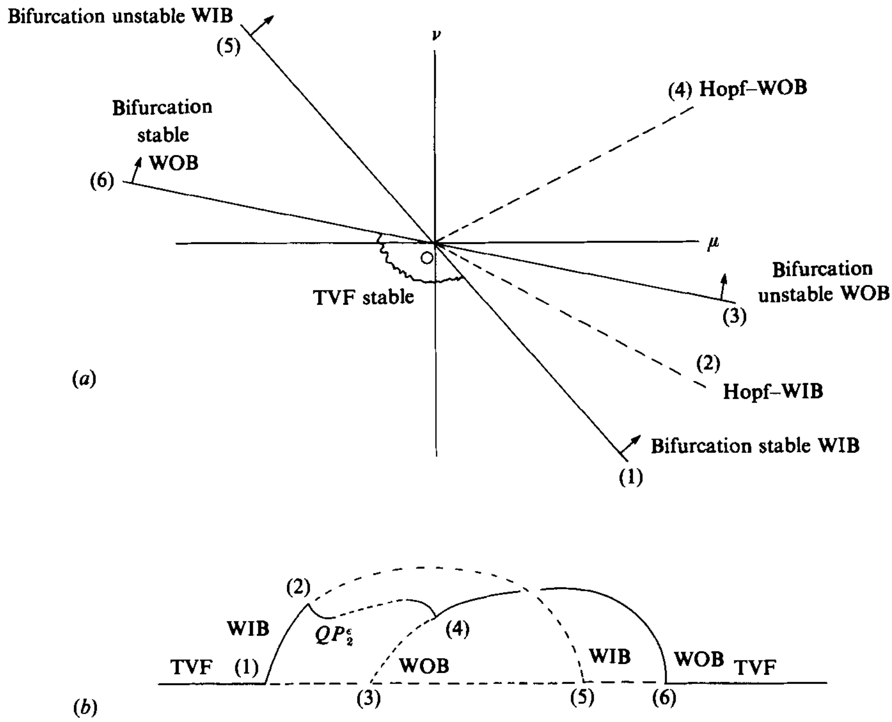

FIGURE 6. The second example of WIB–WOB interaction. Here wavelets QP_2^ϵ are stable when they occur. We cannot affirm that there is a unique branch QP_2^ϵ.

Now, depending on coefficients, which in principle could be computed once the TVF and the eigenmodes are known, there are a lot of possibilities. We give in figures 5 and 6 two different cases. In the first there are values of parameters for which both flows WIB and WOB are stable, i.e. we see one or the other flow depending on initial conditions. The second case gives wavelets (QP_2^ϵ), which could be seen since they are stable. In both cases we choose a_{1r}, a'_{1r} positive, and b_r, c'_r negative (supercritical bifurcations).

The author is very grateful to the referees for suggesting improvements to this paper and especially to the one who found an error in the version presented at the Symposium.

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